

Changes of Variables and the Renormalization Group*

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Abstract

A class of exact infinitesimal renormalization group transformations is proposed and studied. These transformations are pure changes of variables (*i.e.*, no integration or elimination of some degrees of freedom is required) such that a saddle point approximation is more accurate, becoming, in some cases asymptotically exact as the transformations are iterated. The formalism provides a simplified and unified approach to several known renormalization groups. It also suggests some new ways in which renormalization group methods might successfully be applied. In particular, an exact gauge covariant renormalization group transformation is constructed. Solutions for a scalar field theory are obtained both as an expansion in $\varepsilon = 4 - d$ and as an expansion in a single coupling constant.

1 Introduction

For quite some time now it has been apparent that the success of the renormalization group (RG) methods (see *e.g.*, see and references therein) in dealing with problems involving many degrees of freedom is connected to an appropriate choice of variables. That is, the various RG's provide systematic ways to focus one's attention on the degrees of freedom that are most important in the problem under consideration. For example, in Wilson's approach to the problem of critical phenomena [2] it is recognized that short wavelength degrees of freedom are not interesting in themselves, but only indirectly through the effective interactions they induce between the experimentally accessible long wavelength degrees of freedom. The strategy is to eliminate the short wavelengths in a series of steps. One starts by eliminating the shortest ones first, then slightly longer ones, and so on, gradually working one's way towards an effective Lagrangian which contains only the relevant degrees of freedom.

*This work [1] first appeared as Caltech preprint CALT-68-1099. It was supported in part by the Department of Energy under contract DEAC 03-81-ER0050.

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In RG's such as the original Gell-Mann and Low RG [4] the appropriate choice of variables was achieved in quite a different way. The point is that while all wavelengths contribute to a loop integration in a given Feynman diagram, the actual relative contribution of the short versus the long wavelengths depends on the renormalization scale chosen. The freedom to change the renormalization scale thus allows us to emphasize some degrees of freedom over others and therefore to improve the perturbative calculation.

The two approaches above are sufficiently different that in spite of yielding the same results when applied to a given problem the connection between the two has been a matter of some confusion.

The Wilson RG transformation involves not only an elimination of some degrees of freedom but also an explicit change of variables. Some consequences of this fact appeared in works by Jona-Lasinio [5] and Wegner [6]. These authors were concerned with the possibility of choosing more general RG transformations and showing that physically significant quantities such as critical exponents are independent of such a choice. Thus, Jona-Lasinio defines generalized renormalization transformations as all those that leave the effective action Γ invariant in value. It seems unlikely that one can be more general than that, but this evades the important issue of which transformations are useful. Wegner goes further. He recognizes that transformations can be made in a rather general way and makes the essential remark that elimination of degrees of freedom is not a necessary step since some changes of variables effectively accomplish such an elimination. He then goes on to exhibit explicitly the transformation that generates Wilson's incomplete-integration RG [2] and to conjecture that useful transformations would involve some kind of nonlinearity perhaps through some unspecified dependence on the Hamiltonian.

In this work [1] I study a class of exact infinitesimal RG transformations for field theories in continuum space that are pure changes of variables, *i.e.*, no additional elimination or integration of certain degrees of freedom is required. To isolate the minimal structure a change of variables needs to include in order to actually accomplish this, I formulate in section 2 three exact RG's in differential form. These are equivalent though simplified versions of the sharp-cutoff RG of Wegner and Houghton [7], of the incomplete integration RG of Wilson [2], and of the hard-soft splitting RG [8]-[11].

In section 3 the required change of variables is obtained as well as the RG equations both in functional form and as an infinite set of integro-differential equations. The reason why this class of RG's is useful is immediately apparent: the changes of variables are such that a classical or saddle-point approximation in the new variables is more accurate. No mention is made of the question of long versus short wavelengths; this is important. On iterating the RG transformations (*i.e.*, on solving the equations for the RG evolution of the action or of the Hamiltonian) the classical approximation becomes better and better approaching the exact result. Since these RG equations are much simpler than other sets of equations that need to be tackled in order to solve quantum field theories (*e.g.*, the Schwinger-Dyson equations), it suggests that this is a promising way (as is the case with other RG's) to leap beyond the limitations

of perturbation theory.

A further fortunate feature is related to the possibility of applying these RG's to gauge theories. The original motivation for undertaking this study was to construct an exact gauge covariant RG transformation. Such a transformation would allow one to impose the stringent constraints of manifest gauge invariance on the RG-evolved action and perhaps obtain a non-perturbative solution of the RG equations. An analogous program has been partially carried out by Baker, Ball, and Zachariasen (see *e.g.*, [12] and references therein).

Once the structure of changes of variables that are also RG transformations is identified the actual construction of gauge covariant transformations is trivial. The application to gauge theories is discussed in another publication [13]. The calculation of Green's functions is considered in section 4. As with most manipulations with path-integrals the level of mathematical rigor is fairly low. Changes of variables occasionally produce surprises in that the new Lagrangian differs from the one that would be naively obtained. In some situations the additional terms can be cast in the form of an extra potential of order \hbar^2 , in other situations they can be traced to nontrivial Jacobian factors and they generate anomalies (see *e.g.*, [14][15] and references therein). Two explicit solutions of the RG equations in section 5 serve as a check that in our case no such surprises occur. The first solution is an expansion in $\varepsilon = 4 - d$ [2][16], the second is an expansion in a single coupling constant. Both give the same results, but they represent different viewpoints. The former emphasizes analyticity, the latter is closer in spirit to the Gell-Mann and Low spirit. Finally, the conclusions and some comments appear in section 6. Some of the details of the calculations and a pedagogical example, a scalar field theory in zero dimensions (a single integral) are discussed in the appendices.

Note added: In the many years since this paper [1] was written a considerable amount of research has been carried out on the subject which is now variously known as the exact RG, the functional RG, and the non-perturbative RG. Much of the material presented here has been independently rediscovered, and there have been important developments that go far beyond the original scope of this paper. Prominent among the latter is the work by J. Polchinski [17] where exact RG methods were developed as a method to prove renormalizability. The computational power of the exact RG has been extended in the work by C. Wetterich [18] and coauthors — the effective average action method — including its application to Yang-Mills theory [19] and gravity [20]. Wegner's original insight of the RG transformation as a change of variables has been considerably expanded by T. R. Morris and co-workers [21][22]. (For additional references see the excellent reviews [23]-[29].) Despite such extensive literature the point of view presented here may still have some pedagogical value since some of our results — the interpretation of the RG as a change of variables that systematically improves a saddle point approximation, the calculation of the RG flow of Green's functions, and the toy model of a field theory in zero dimensions — do not seem to have yet appeared in print.

2 Three exact differential RG's

2.1 Sharp-Cutoff RG

Here I present a modified version of the sharp-cutoff RG derived by Wegner and Houghton [7]. Consider the Green's function generating functional in Euclidean space,

$$Z = \int D\phi \exp -S_\tau[\phi] . \quad (1)$$

For the moment I will not be concerned with coupling the field ϕ to external sources. This problem will be addressed in section 4.

Suppose field components with momenta larger than a certain cutoff $\Lambda_\tau = \Lambda e^{-\tau}$ have been integrated out, *i.e.*,

$$\phi(q) = 0 \quad \text{for } q > \Lambda_\tau . \quad (2)$$

This implies that the effective action S_τ consists not just of the simple interactions contained in the bare action $S = S_{-\infty}$ but rather of an infinite number of arbitrarily complicated interactions. Our problem is to study how the action evolves when the cutoff is slightly decreased to $\Lambda_{\tau+\delta\tau}$. Suppose we separate out the field components with momenta in the thin shell between $\Lambda_{\tau+\delta\tau}$ and Λ_τ ,

$$\phi(x) \rightarrow \phi(x) + \sigma(x) , \quad (3)$$

where on the right hand side $\phi(q) = 0$ for $q > \Lambda_{\tau+\delta\tau}$ and $\sigma(q) = 0$ for q outside the thin shell of thickness $\Lambda_\tau \delta\tau$. On integrating out the σ fields the new action will be given by

$$\exp -S_{\tau+\delta\tau}[\phi] = \int D\sigma \exp -S_\tau[\phi + \sigma] . \quad (4)$$

The integration is performed perturbatively in three steps: first expand $S_\tau[\phi + \sigma]$ in a power series in σ ; second, isolate a convenient σ -field propagator; and third, treat the remaining σ^p vertices ($p \geq 1$) as a perturbation. This procedure, carried out in detail in Appendix A shows that to first order in the shell thickness only diagrams with one internal σ -field line contribute. The result is

$$S_{\tau+\delta\tau}[\phi] - S_\tau[\phi] = \int d^d q (2\pi)^d \Delta_\tau(q) \left[\frac{\delta^2 S_\tau}{\delta\phi(q)\delta\phi(-q)} - \frac{\delta S_\tau}{\delta\phi(q)} \frac{\delta S_\tau}{\delta\phi(-q)} \right] , \quad (5)$$

where $\Delta_\tau(q)$ is a convenient σ propagator which vanishes outside the shell. Alternatively,

$$S_{\tau+\delta\tau}[\phi] - S_\tau[\phi] = (2\pi)^d \Lambda_\tau^{d-2} \delta\tau \int d\Omega_d \left[\frac{\delta^2 S_\tau}{\delta\phi(q)\delta\phi(-q)} - \frac{\delta S_\tau}{\delta\phi(q)} \frac{\delta S_\tau}{\delta\phi(-q)} \right] , \quad (6)$$

where now $q^2 = \Lambda_\tau^2$ and $d\Omega_d$ is the element of solid angle in d dimensions.

To obtain RG equations an additional dilatation change of variables is required. This is a trivial step which will be addressed later in section 3.

The basic improvement of eqs.(5) and (6) over those of Wegner and Houghton is that their equation include all diagrams with one internal σ -field loop (*i.e.*, many internal σ propagators) while ours include only the much smaller set of diagrams with only one σ -field propagator.

2.2 Incomplete-integration RG

In order to avoid the unphysical difficulties introduced by the discontinuous cutoff considered in the last section, Wilson [2] introduced the concept of incomplete integration designed to achieve a smooth interpolation between those degrees of freedom that have been integrated out and those that have not.

The idea is implemented through the introduction of an auxiliary functional $\delta_\alpha[\phi]$. In the case of an ordinary single integral $\delta_\alpha(x)$ is a function such that as α goes from 0 to ∞ , the function

$$z_\alpha(x) = \int dy z_0(y) \delta_\alpha(y - x)$$

smoothly interpolates between the integrand $z_0(y)$ and the integral

$$z_\infty(x) = \int dy z_0(y) .$$

All that is required is that

$$\delta_\alpha(x) \rightarrow \begin{cases} \delta(x) & \text{for } \alpha \rightarrow 0 , \\ 1 & \text{for } \alpha \rightarrow \infty . \end{cases}$$

Wilson's choice for δ_α was the Green's function of a certain differential equation. It is perhaps simpler to use a Gaussian,

$$\delta_\alpha(x) = \left(\frac{1}{4\pi\alpha} + 1 \right)^{1/2} \exp -\frac{x^2}{4\alpha} .$$

The case of a single integral is pursued further in appendix B. now we return to the functional integral problem. We let $\alpha = \alpha(q, \tau)$ and introduce

$$\text{const.} = \int D\phi \delta_\alpha[\phi - \Phi]$$

into the functional integral

$$Z = \int D\Phi \exp -S[\Phi] .$$

The result is

$$Z = \int D\phi \exp -S_\tau[\phi] ,$$

where

$$\begin{aligned}\exp -S_\tau[\phi] &= \int D\Phi \delta_\alpha[\phi - \Phi] \exp -S[\Phi] \\ &= \int D\Phi \exp - \left(S[\Phi] + \int d\tilde{q} \frac{|\phi(q) - \Phi(q)|^2}{4\alpha(q, \tau)} \right) ,\end{aligned}\quad (7)$$

where we drop an unimportant field-independent factor. In eq.(7) and in the following we adopt the notation

$$d\tilde{q} = \frac{d^d q}{(2\pi)^d} \quad \text{and} \quad \tilde{\delta}(q) = (2\pi)^d \delta^d(q) .$$

The conventional usage is to choose $\alpha(q, \tau)$ such that $S_\tau[\phi]$ describes modes with an effective cutoff $\Lambda_\tau = \Lambda e^{-\tau}$ which means that $\alpha(q, \tau)$ is very large for $q \gg \Lambda_\tau$ and very small for $q \ll \Lambda_\tau$. A convenient, but by no means obligatory choice is one in which the mode $\phi(qe^{-\delta\tau})$ in $S_{\tau+\delta\tau}$ is integrated out to the same extent as the mode $\phi(q)$ in S_τ , *i.e.*,

$$\alpha(qe^{-\delta\tau}, \tau + \delta\tau) = \alpha(q, \tau) .$$

This implies that α depends on q and τ only through the combination qe^τ . Let

$$\alpha_\tau(q) = \alpha(q/\Lambda_\tau) . \quad (8)$$

The functional integral (7) can be transformed into a functional differential equation,

$$\frac{d}{d\tau} S_\tau = \int d^d q (2\pi)^d \dot{\alpha}_\tau(q) \left[\frac{\delta^2 S_\tau}{\delta\phi(q)\delta\phi(-q)} - \frac{\delta S_\tau}{\delta\phi(q)} \frac{\delta S_\tau}{\delta\phi(-q)} \right] \quad (9)$$

where $\dot{\alpha}_\tau = d\alpha_\tau/d\tau$. This equation is obtained noticing that differentiation of (7) with respect to τ brings down factors of $(\phi - \Phi)$ on the right hand side that may also be brought through functional differentiation with respect to ϕ . Comparison of eq.(9) with the remarkably similar eq.(5) shows that $\dot{\alpha}_\tau d\tau$ is playing the role of a propagator.

Again, the full RG equations require an additional dilation which we postpone until section 3.

2.3 Hard-Soft Splitting RG

Another method which allows elimination of short wavelength degrees of freedom was suggested by Wilson [8]. It consists of splitting the propagator into two pieces, one contributes dominantly for high momenta while the other does so for low momenta,

$$\frac{1}{p^2} = \frac{1}{p^2 + \mu^2} + \frac{\mu^2}{p^2(p^2 + \mu^2)} . \quad (10)$$

The idea was to take advantage of the UV asymptotic freedom of Yang-Mills theories to integrate out the hard components in renormalized perturbation theory to generate an effective action for the soft components which could be treated using techniques better suited to the strong coupling regime.

The method was further developed by Lowenstein and Mitter [9] and by Mitter and Valent [10], and applied to the weak coupling regime of quantum chromodynamics by Shalloway [11]. In this section we formulate it in a simple way which allows immediate comparison with the RG's described in the previous sections.

The actual splitting of hard and soft components is accomplished by introducing

$$\text{const.} = \int D\chi \exp - \int dx \frac{1}{2} \mu^2 \chi^2$$

into the path integral

$$Z = \int D\Phi \exp - S[\Phi] \quad \text{where} \quad S[\Phi] = \int dx \left[\frac{1}{2} \partial\Phi\partial\Phi + V(\Phi) \right],$$

and then making the change of variables $(\Phi, \chi) \rightarrow (\phi, \phi_h)$ where

$$\Phi = \phi + \phi_h \quad \text{and} \quad \chi = \phi_h + \frac{\partial^2}{\mu^2} \phi.$$

The result is

$$Z = \int D\phi D\phi_h \exp \int dx \left[\frac{1}{2} \phi \partial^2 \left(1 - \frac{\partial^2}{\mu^2} \right) \phi + \frac{1}{2} \phi_h (\partial^2 - \mu^2) \phi_h - V(\phi + \phi_h) \right],$$

where the hard-soft separation (10) is explicit. Integrating over ϕ_h leads once more to

$$Z = \int D\phi \exp - S_\tau[\phi],$$

where

$$\exp - S_\tau[\phi] = \int D\phi_h \exp - \left[S[\phi + \phi_h] + \int dx \frac{1}{2} \mu^2 \left(\phi_h + \frac{\partial^2}{\mu^2} \phi \right)^2 \right].$$

We wish to study the evolution of S_τ under changes of τ . We are taking $\mu = \mu(\tau)$. It is convenient to shift integration variables back to Φ . Then

$$\exp - S_\tau[\phi] = \int D\Phi \exp - \left[S[\Phi] + \int d\tilde{q} \frac{1}{2} \mu^2 (\rho\phi(q) - \Phi(q))^2 \right]$$

where $\rho = 1 + q^2/\mu^2$. This equation, which is very similar to (7), can also be transformed into a functional differential equation,

$$\frac{dS_\tau}{d\tau} = \int \frac{d^d q}{2\rho^2} \left[(2\pi)^d \frac{d\mu^{-2}}{d\tau} \left(\frac{\delta^2 S_\tau}{\delta\phi(q)\delta\phi(-q)} - \frac{\delta S_\tau}{\delta\phi(q)} \frac{\delta S_\tau}{\delta\phi(-q)} \right) + \frac{d\rho^2}{d\tau} \phi(q) \frac{\delta S_\tau}{\delta\phi(q)} \right]. \quad (11)$$

This differs from eq.(5) and 7 only in the last term, which by now one might suspect is not essential.

Incidentally, one could consider situations where μ and ρ have momentum dependencies other than those assumed above, in particular one could choose

$$(2\pi)^d \mu^{-2} = \rho^2 + c \quad (12)$$

where c is independent of τ . Then eq.(11) becomes identical with the original incomplete-integration equation of Wilson [eq.(11.8) of ref.[2]].

The important conclusion to be drawn is that the various examples of exact RG's considered above are characterized by a certain common feature, which we might guess is what makes them useful RG's in the first place. The variations are presumably inessential in principle, although in practice they may be important. For example, the sharp-cutoff RG is definitely more inconvenient to calculate with.

3 The RG as a change of variables

Functional integrals are a particularly convenient way to formulate quantum field theories not just because they readily allow for perturbative expansions but also because the implications of invariances and of the changes induced by transformations of the dynamical variables can be easily studied. This feature has been found particularly useful in the case of non-Abelian gauge transformations. In this section we consider infinitesimal variable changes that reproduce the exact RG transformations described previously.

Let us go back to eq.(1) and investigate the changes in the action S_τ induced by the variable transformation

$$\phi(q) \rightarrow \phi'(q) = \phi(q) + \delta\tau \eta_\tau[\phi, q] , \quad (13)$$

where $\eta_\tau[\phi, q]$ is some sufficiently well-behaved functional of ϕ and a function of q . Taking into account the Jacobian of this transformation eq.(1) becomes,

$$\begin{aligned} Z &= \int D\phi \left[1 + \delta\tau \int dq \frac{\delta\eta_\tau[\phi, q]}{\delta\phi(q)} \right] \exp - \left[S_\tau[\phi] + \delta\tau \int dq \frac{\delta S_\tau}{\delta\phi(q)} \eta_\tau[\phi, q] \right] \\ &= \int D\phi \exp - S_{\tau+\delta\tau}[\phi] , \end{aligned}$$

where

$$S_{\tau+\delta\tau}[\phi] = S_\tau[\phi] + \delta\tau \int dq \left[\frac{\delta S_\tau}{\delta\phi(q)} \eta_\tau[\phi, q] - \frac{\delta\eta_\tau[\phi, q]}{\delta\phi(q)} \right] . \quad (14)$$

Suppose one chooses

$$\eta_\tau[\phi, q] = -(2\pi)^d \dot{\alpha}_\tau(q) \frac{\delta S_\tau}{\delta\phi(-q)} , \quad (15)$$

then eq.(14) becomes identical to the Gaussian incomplete-integration RG. More generally, if one also includes an inessential rescaling of the field,

$$\eta_\tau[\phi, q] = -(2\pi)^d \dot{\alpha}_\tau(q) \frac{\delta S_\tau}{\delta \phi(-q)} + \zeta_\tau(q) \phi(q) ,$$

one obtains an equation of the form of eq.(11).

The conclusion is that transformations of the form of eq.(15) are RG transformations.

Furthermore, one can see why they are useful transformations. A field configuration that is a solution of the classical equation of motion $\delta S_\tau / \delta \phi = 0$ will not be affected by (15). Any other configuration will flow with τ until it becomes a classical solution (*i.e.*, a stationary point), then it ceases to flow. As $\tau \rightarrow \infty$ a situation is approached in which all field configurations are classical solutions, *i.e.*, $\delta S_\tau / \delta \phi = 0$ for all ϕ . The action approaches a constant.

In appendix B a toy example in zero spacetime dimensions, an ordinary integral, is worked out. It allows one to see very clearly what is happening. The changes of variables are such that a “classical” approximation, *i.e.*, a steepest descent approximation becomes better and better as τ increases, approaching the exact result as $\tau \rightarrow \infty$. The reason the approximation is improved is not that the integrand becomes steeper as one might at first guess, but rather that it approaches a Gaussian for which the steepest descent method is exact. The fact that this Gaussian is increasingly flatter (the action becomes a constant) is not a serious obstacle. It merely requires us to calculate the integral before the limit $\tau = 0$ is reached.

Traditionally RG techniques have been applied to problems that exhibit some kind of symmetry under changes of scale. In these cases it is convenient to perform an additional change of variables in the form of a dilatation. Consider a situation in which the effective cutoff is Λ . Under the change

$$\phi(q) \rightarrow \phi(q) + \delta\tau \eta_0[\phi, q] \quad \text{where} \quad \eta_0[\phi, q] = \eta_\tau[\phi, q]|_{\tau=0}$$

the effective cutoff is changed to $\Lambda e^{-\delta\tau}$. The scaling transformation $q \rightarrow q e^{-\delta\tau}$ then guarantees that the new momenta will span the same range $(0, \Lambda)$ as before. Thus one takes

$$\delta_{\text{dil}}\phi(q) = \delta\tau \left(d - d_\phi + q \cdot \frac{\partial}{\partial q} \right) \phi(q) ,$$

where the field scale dimension,

$$d_\phi = \frac{d}{2} - 1 + \gamma_\phi ,$$

includes an anomalous dimension term.

The full RG transformation is

$$\phi(q) \rightarrow \phi(q) + \delta\tau \eta_0[\phi, q] + \delta\tau \left(d - d_\phi + q \cdot \frac{\partial}{\partial q} \right) \phi(q) , \quad (16)$$

and the full RG equation is

$$\begin{aligned} \frac{d}{d\tau} S_\tau = \int d^d q (2\pi)^d \dot{\alpha}(q) & \left[\frac{\delta^2 S_\tau}{\delta \phi(q) \delta \phi(-q)} - \frac{\delta S_\tau}{\delta \phi(q)} \frac{\delta S_\tau}{\delta \phi(-q)} \right] \\ & + \frac{\delta S_\tau}{\delta \phi(q)} \left(d - d_\phi + q \cdot \frac{\partial}{\partial q} \right) \phi(q) , \end{aligned} \quad (17)$$

where now $\dot{\alpha} = d\alpha/d\tau|_{\tau=0}$.

The functional equation (17) can be transformed into an infinite set of integro-differential equations. Substituting an action of the general form

$$S_\tau[\phi] = \sum_{n \text{ even}}^{\infty} \frac{1}{n!} \int d\tilde{q}_1 \dots d\tilde{q}_n \tilde{\delta} \left(\sum_j^n q_j \right) u_n(q_1 \dots q_n, \tau) \phi(q_1) \dots \phi(q_n) \quad (18)$$

into eq.(17) and equating the coefficients of terms of the same degree in ϕ one obtains (omitting the τ dependence)

$$\begin{aligned} \frac{\partial}{\partial \tau} u_n(q_1 \dots q_n) &= \int d\tilde{k} \dot{\alpha}(k) u_{n+2}(q_1 \dots q_n, k, -k) \\ &+ \sum_{m \geq 2}^n \binom{n}{m-1} \frac{1}{n!} \sum_{\{q_j\}} \dot{\alpha}(k_m) u_m(q_1 \dots q_{m-1}, k_m) u_{n-m+2}(q_m \dots q_n, -k_m) \\ &+ \left[d - nd_\phi - \sum_{j=1}^n q_j \cdot \frac{\partial}{\partial q_j} \right] u_n(q_1 \dots q_n) , \end{aligned} \quad (19)$$

where

$$k_m = - \sum_{j=1}^{m-1} q_j = \sum_{j=m}^n q_j$$

and where $\sum_{\{q_j\}}$ denotes a sum over all the permutations of the q_j 's.

Equations (19) can be given a simple graphical representation in which the first term on the right hand side is a loop diagram with $\dot{\alpha}(k)$ playing the role of the internal line propagator; the second term is a tree diagram where $\dot{\alpha}(k_m)$ is again the propagator for the internal line. When a sharp-cutoff is employed, as discussed in section 2.2, the $\dot{\alpha}$'s do actually correspond to propagators in the conventional sense.

4 Green's Functions

The calculation of Green's functions or of correlation functions brings us to the problem of coupling the field ϕ to an external source. The study of how Green's functions calculated from the bare action S are related to those calculated from S_τ can be done in a number of ways (see *e.g.* [2]). We would like to address this question in the spirit of the previous section, regarding the RG as an infinitesimal change of variables.

Consider the generating functional

$$Z_\tau[j] = \int D\phi \exp \left(-S_\tau[\phi] + \int j\phi \right) .$$

Performing a change of variables of the form of eq.(14) (for simplicity we do not include the dilatation change of variables) we obtain,

$$Z_\tau[j] = \int D\phi \left[1 - \delta\tau \int d^d q j(-q) \dot{\alpha}_\tau(q) \frac{\delta S_\tau}{\delta \phi(-q)} \right] \exp \left(-S_\tau[\phi] + \int j\phi \right) .$$

But,

$$0 = \int D\phi \frac{\delta}{\delta \phi(-q)} e^{-S_\tau[\phi] + \int j\phi} = \int D\phi \left[\frac{j(q)}{(2\pi)^d} - \frac{\delta S_\tau}{\delta \phi(-q)} \right] e^{-S_\tau[\phi] + \int j\phi} ,$$

and therefore

$$\frac{dZ_\tau}{d\tau} = \left[\int d\tilde{q} j(-q) \dot{\alpha}_\tau(q) j(q) \right] Z_\tau .$$

Integrating in τ with the initial condition $S_{-\infty} = S$, *i.e.*,

$$Z_{-\infty}[j] = Z[j] = \int D\phi \exp \left(-S[\phi] + \int j\phi \right) ,$$

leads to

$$Z[j] = \left[\int d\tilde{q} j(-q) \alpha_\tau(q) j(q) \right] Z_\tau[j] ,$$

which exhibits the desired relation in a particularly simple form.

The generating functional of connected Green's functions, $W = -\log Z$, and the corresponding W_τ are related by

$$W[j] = W_\tau[j] + \int d\tilde{q} j(-q) \alpha_\tau(q) j(q) .$$

This shows that the connected n -point functions computed with S_τ are identical with those computed with S for $n \geq 3$ while for $n = 2$ they differ in a rather trivial way. In this formulation it is then particularly clear that physically significant quantities such as critical exponents or S -matrix elements can be computed with either Z or Z_τ and that they are independent of the choice of α , that is, independent of the choice of the RG.

5 Solutions

Obtaining solutions to the RG equations (19) is a challenging problem. In this section two conventional approximations are discussed, an expansion in $\varepsilon = 4-d$ and an expansion in a single coupling constant. One motivation is to give us confidence that the expressions in the previous sections are correct in spite of

the lack of mathematical rigor employed in their deduction. Another motivation is to compare two approximation schemes which, although leading to differential equations of similar structure, represent different viewpoints. Finally, yet a third motivation is to stress the larger freedom of choice of the RG. This is important, not only because it allows one to construct RG's in which the usual restriction of integrating only over the short wavelength degrees of freedom is lifted, but also because it will allow us to construct gauge covariant RG's.

A standard approach to solving eq.(19) consists of finding a fixed point and studying the evolution of small perturbations about this fixed point. One looks for a fixed point solution S^* for which the vertex functions u_n^* do not depend on τ as an expansion in ε ,

$$\begin{aligned} u_2^* &= V_{20} + V_{21}\varepsilon + V_{22}\varepsilon^2 + \dots \\ u_4^* &= V_{41}\varepsilon + V_{42}\varepsilon^2 + \dots \\ u_6^* &= V_{62}\varepsilon^2 + \dots \end{aligned} \tag{20}$$

with the anomalous dimension given by

$$\gamma_\phi = \gamma_1\varepsilon + \gamma_2\varepsilon^2 + \dots \tag{21}$$

The crucial extra condition imposed on the solution and on the RG transformation (16) (*i.e.*, on the anomalous dimension γ_ϕ) is that the solution be analytic in the momenta. The need for this condition can be vaguely argued as follows. Non-analyticity in momentum space translates into long range of nonlocal interactions in position space for which some features of critical behavior, like universality, are known not to hold.

Details of these calculations, which are similar to those obtained in [16] for Wilson's incomplete-integration RG, can be found in appendix C.

An alternative perturbative approach consists in expanding in a single coupling constant $g(\tau)$ without referring to any fixed point. One looks for a solution of the form

$$\begin{aligned} u_2 &= U_{20} + gU_{41} + g^2U_{42} + \dots \\ u_4 &= \Lambda^\varepsilon(gV_{41} + g^2U_{42} + \dots) \\ u_6 &= \Lambda^{2\varepsilon}(g^2U_{62} + \dots) , \end{aligned} \tag{22}$$

with the anomalous dimension given by

$$\gamma_\phi = \gamma_1g + \gamma_2g^2 + \dots \tag{23}$$

and $g(\tau)$ flowing according to

$$\frac{dg}{d\tau} = -\beta(g) = b_1g + b_2g^2 + \dots \tag{24}$$

Factors of Λ^ε have been made explicit so that various U 's have the same dimensions they would have in $d = 4$. The crucial extra condition imposed on the

solution and on the RG transformation in this perturbative approach is that all τ dependence be contained in the single function $g(\tau)$. This brings us somewhat closer to the spirit of the RG of Gell-Mann and Low. The other functions are required not to depend on τ but could be non-analytic. These calculations are carried out in appendix D.

While none of the results obtained in those calculations are new the freedom of the choice of the cutoff function $\dot{\alpha}(q)$ is explicit. In particular, one is not required to integrate only short wavelengths, that is $\dot{\alpha}(0) = 0$, as for example in the usual choice $\dot{\alpha}(q) = q^2/\Lambda^4$. One can integrate the long wavelengths as well, for example $\dot{\alpha}(q) = \Lambda^{-2} \exp(q^2/\Lambda^2)$ or even integrate all wavelengths simultaneously to the same extent by taking $\dot{\alpha} = \Lambda^{-2} = \text{const.}$

Although physically significant quantities such as γ_ϕ or $\beta(g)$ are independent of $\dot{\alpha}$ the same is not true of the vertex functions u_n . In particular one should be careful with the other wise very convenient choice $\dot{\alpha} = \text{const.}$ For this choice of $\dot{\alpha}$ the vertex function contain parts that are divergent as $d \rightarrow 4$. This is annoying but nothing more. The way around this problem is the usual one, to think of the u_n 's as separated into two parts $u_n = u_n^R + u_n^C$ one of which is finite while the other is a divergent counterterm. The RG equations (19) keep track of the evolution of both the finite and the divergent parts. The presence of these divergences is a manifestation of the fact that while the RG was historically connected to renormalization theory, such a connection, although sometimes convenient, is not at all necessary.

6 Conclusions

In this work an approach to the renormalization group has been developed in which the RG transformations are convenient changes of variables. The main conclusions of this work are enumerated below.

1. A class of exact infinitesimal RG transformations has been proposed. The form of the transformations is suggested quite naturally after several known exact RG's are formulated in a conveniently simplified way. Conversely, those exact RG's can be treated as special cases of a more general formalism.
2. The transformations are pure changes of variables (*i.e.*, no explicit integration or elimination of some degrees of freedom is required) such that a saddle point approximations is more accurate, becoming in some cases, asymptotically exact as the transformations are iterated.
3. Solutions of the RG equations for a scalar field theory were obtained both as an expansion in $\varepsilon = 4 - d$ and as an expansion in a single coupling constant. Physically significant results agree with those obtained following conventional methods. The well-known fact that physical quantities such as critical exponents are independent of the particular RG employed emerges quite clearly.

The consideration of RG's from this generalized point of view has a number of attractive features which immediately suggest many possible applications. For example, the method could be extended to any problem where a saddle approximation is used. One could perhaps obtain improved large N expansions.

The role of dilatations is de-emphasized and one might profitably attack problems where the issue is not the symmetry under scale transformations or its breaking. It should be possible to study the phenomena of dynamical symmetry breaking or of dynamical symmetry restoration. The localization of the minima of the classical action S and of the RG-evolved action S_τ need not coincide and it is the latter that will give more reliable information about the true minima. Another related possible application could be in the study of anomalies. Again, the true symmetry of a quantum theory could be more reliably established by classically studying the RG-evolved action S_τ , which includes some quantum effects, than by classically studying the action S .

A further attractive feature is that the exact RG's formulated above do not require the successive elimination of certain degrees of freedom and can therefore be applied to systems with a small number or even just one degree of freedom (see appendix B). They can also be exactly applied to field theories defined on a lattice (see [13]).

Finally, although for simplicity we have only dealt with scalar field theories symmetric under $\phi \rightarrow -\phi$ the extension to other fields, fermions, etc. is straightforward. As mentioned in the Introduction the original motivation for this study was to construct a gauge covariant RG transformation. Once one establishes that changes of variables of the form

$$\phi(x) \rightarrow \phi(x) - \delta\tau \dot{\alpha}_\tau (-i\partial) \frac{\delta S_\tau[\phi]}{\delta\phi(x)} ,$$

are indeed RG transformations, the problem of gauge covariance is easily solved by replacing ordinary derivatives by covariant derivatives,

$$A(x) \rightarrow A(x) - \delta\tau \dot{\alpha}_\tau (-iD) \frac{\delta S_\tau[A]}{\delta A(x)} . \quad (25)$$

Perhaps the simplest choice is $\dot{\alpha}_\tau = \text{const}$. A detailed study of the application of this kind of RG to non-Abelian gauge theories is the subject of a companion article [13].

Acknowledgments: I would like to thank Professor F. Zachariasen and very especially Néstor Caticha for many stimulating discussions and encouragement.

Appendix A. The Sharp-Cutoff RG

In this appendix we integrate out the σ fields in the thin momentum shell and deduce eqs.(5) and (6). As discussed in section 2.1 this process involves three steps.

Step 1: Expand $S_\tau[\phi + \sigma]$ in a power series about $\sigma = 0$,

$$S_\tau[\phi + \sigma] = \sum_{p=0}^{\infty} \frac{1}{p!} \int dx_1 \dots dx_p S_\tau^{(p)}[\phi; x_1 \dots x_p] \sigma(x_1) \dots \sigma(x_p) , \quad (\text{A.1})$$

where

$$S_\tau^{(p)}[\phi; x_1 \dots x_p] = \frac{\delta^p S_\tau[\phi]}{\delta\phi(x_1) \dots \delta\phi(x_p)} .$$

Step 2: Identify a convenient σ -field propagator. Rewrite the quadratic term in eq.(A.1) as

$$\begin{aligned} \frac{1}{2} \int dx_1 dx_2 S_\tau^{(2)}[\phi; x_1 x_2] \sigma(x_1) \sigma(x_2) &= \frac{1}{2} \int dx \sigma(x) \partial^2 \sigma(x) + \\ &+ \frac{1}{2} \int dx_1 dx_2 \bar{S}_\tau^{(2)}[\phi; x_1 x_2] , \end{aligned}$$

and let $S_\tau^{(p)} = \bar{S}_\tau^{(p)}$ for $p \neq 2$, so that a convenient propagator is

$$\Delta_\tau(x - y) = \int d\tilde{q} \Delta_\tau(q) e^{-iq(x-y)} = \frac{\delta\tau \Lambda_\tau^{d-2}}{(2\pi)^d} \int d\Omega_d e^{-iq(x-y)} . \quad (\text{A.2})$$

where $d\tilde{q} = d^d q / (2\pi)^d$.

Step 3: Treat the σ^p vertices perturbatively. Rewrite eq.(4) in the form

$$\begin{aligned} \exp -S_{\tau+\delta\tau}[\phi] &= \exp - \left[\sum_{p=0}^{\infty} \frac{1}{p!} \int dx_1 \dots dx_p \bar{S}_\tau^{(p)} \frac{\delta}{\delta j(x_1)} \dots \frac{\delta}{\delta j(x_p)} \right] \\ &\int D\sigma \exp - \int dx \left(\frac{1}{2} \sigma \Delta_\tau^{-1} \sigma - j\sigma \right) \Big|_{j=0} . \end{aligned}$$

Since each σ propagator contributes a factor of $\delta\tau$, eq.(A.2), while each vertex $\bar{S}_\tau^{(p)}$ contributes a factor $(\delta\tau)^0$ it follows that to first order in $\delta\tau$ only diagrams with one internal σ line need be included. Therefore

$$S_{\tau+\delta\tau}[\phi] = \int dx_1 dx_2 \frac{1}{2} \Delta_\tau(x_1 - x_2) \left[S_\tau^{(2)}(x_1, x_2) - S_\tau^{(1)}(x_1) S_\tau^{(1)}(x_2) \right] + O(\delta\tau^2) ,$$

where we have dropped the bars which amounts to ignoring a ϕ -independent constant. Rewriting this expression in momentum space leads to eq.(5) as desired.

Appendix B. A Field Theory in Zero Dimensions

In this appendix we consider in more detail the application of the RG formalism described previously to a field theory in zero dimensions for which the partition function is an ordinary integral. This study serves to clarify the concepts

in a much simpler setting exhibiting the essence of the RG transformation as a change of variables better suited for a semiclassical approximation, and also to illustrate a point mentioned in section 6, namely that these RG's are not restricted to systems with an infinite number of degrees of freedom. First we deduce the incomplete-integration RG equation interpreting it as a change of variables and then show that a steepest descent approximation becomes asymptotically exact for the RG evolved action. Finally, as a practical example we perform an RG improved perturbative calculation.

As discussed in section 2.2 “incomplete integration” is achieved through the introduction of a constant,

$$1 = N_\alpha^{-1} \int_{-\infty}^{\infty} dy \delta_\alpha(y - x) \quad \text{where} \quad N_\alpha = (1 + 4\pi\alpha)^{1/2}$$

into the “partition function”,

$$z = \int_{-\infty}^{\infty} dy \exp -S(y) = N_\alpha^{-1} \int_{-\infty}^{\infty} dx \exp -S_\alpha(x) \quad (\text{B.1})$$

where

$$\exp -S_\alpha(x) = \int_{-\infty}^{\infty} dy \delta_\alpha(y - x) \exp -S(y) . \quad (\text{B.2})$$

Using

$$\frac{d\delta_\alpha(x)}{d\alpha} = \frac{2\pi\delta_\alpha(x)}{1 + 4\pi} + \frac{d^2\delta_\alpha(x)}{d\alpha^2} ,$$

eq.(B.2) can be turned into an RG differential equation,

$$\frac{dS_\alpha(x)}{d\alpha} = \frac{d^2S_\alpha(x)}{d\alpha^2} - \left[\frac{dS_\alpha(x)}{dx} \right]^2 - \frac{2\pi}{1 + 4\pi} . \quad (\text{B.3})$$

This evolution can be interpreted as a sequence of changes of variables. Changing x to $x' = x + \eta(x)d\alpha$ in (B.1) leads to

$$z = N_{\alpha+d\alpha}^{-1} \int_{-\infty}^{\infty} dx \exp -S_{\alpha+d\alpha}(x) ,$$

where

$$S_{\alpha+d\alpha}(x) = S_\alpha(x) + \frac{dS_\alpha(x)}{dx} \eta d\alpha - \frac{d\eta}{dx} d\alpha - \frac{2\pi d\alpha}{1 + 4\pi} .$$

If one chooses $\eta(x) = -dS_\alpha/dx$ this is precisely the RG equation (B.3).

Equation (B.3) can be transformed into a system of differential equations for the evolution of the “vertex functions”. Substituting

$$S_\alpha(x) = \sum_{n=0, \text{ even}} \frac{1}{n!} u_n(\alpha) x^n ,$$

into (B.3) one obtains

$$\frac{du_0}{d\alpha} = u_2 - \frac{2\pi}{1 + 4\pi} , \quad (\text{B.4a})$$

and for $n \neq 0$,

$$\frac{du_n}{d\alpha} = u_{n+2} - \sum_{m=2}^n \binom{n}{m-1} u_m u_{n-m+2} . \quad (\text{B.4b})$$

Next we come to the question of why it is useful to go through the trouble of solving (B.3). Consider a steepest descent approximation to (B.1):

$$z = \frac{1}{N_\alpha} \left[\frac{2\pi}{S_\alpha^{(2)}(\bar{x}_\alpha)} \right]^{1/2} \exp -S_\alpha(\bar{x}_\alpha) + \dots$$

where $S_\alpha^{(n)}(\bar{x}_\alpha)$ is the n -th derivative of S_α and \bar{x}_α is the saddle point, $S_\alpha^{(1)}(\bar{x}_\alpha) = 0$. The incomplete integration was designed so that as $\alpha \rightarrow \infty$ the exponential factor on the right hand side, $\exp(-S_\alpha)$, tends to the desired exact value z . If the leading steepest descent approximation is to become exact it should be true that

$$\lim_{\alpha \rightarrow \infty} \frac{1}{N_\alpha} \left[\frac{2\pi}{S_\alpha^{(2)}(\bar{x}_\alpha)} \right]^{1/2} = 1 . \quad (\text{B.5})$$

It is easy to see that this is so by referring back to eq.(B.3). As $\alpha \rightarrow \infty$ the left hand side vanishes. Evaluating at the saddle point \bar{x}_α , the second term on the right also vanishes and one gets,

$$S_\alpha^{(2)}(\bar{x}_\alpha) \approx \frac{2\pi}{1 + 4\pi} ,$$

which implies (B.5) as desired.

It is interesting to see what is happening from another point of view. Consider evaluating

$$z = \int_{-\infty}^{\infty} dx \exp - \left(\frac{1}{2} w x^2 + \frac{1}{4!} \lambda x^4 \right) . \quad (\text{B.7})$$

If λ is small enough one could try a perturbative expansion,

$$z \approx \left(\frac{2\pi}{w} \right)^{1/2} \left(1 - \frac{\lambda}{8w^2} + O(\lambda^2) \right) . \quad (\text{B.8})$$

But one could refer to eq.(B.1) and try an RG-improved expansion,

$$z \approx \frac{1}{N_\alpha} e^{u_0(\alpha)} \left(\frac{2\pi}{u_2(\alpha)} \right)^{1/2} \left(1 - \frac{1}{8} \frac{u_4(\alpha)}{u_2^2(\alpha)} + \dots \right) , \quad (\text{B.9})$$

where $u_0(\alpha)$, $u_2(\alpha)$, and $u_4(\alpha)$ are solutions of (B.4) with the initial conditions $u_0(0) = 0$, $u_2(0) = w$, and $u_4(0) = \lambda$. To order λ these solutions are

$$u_0(\alpha) = \frac{1}{2} \log \frac{1 + 2\pi\alpha}{1 + 4\pi\alpha} + \frac{\lambda\alpha^2}{2(1 + 2w\alpha)^2} + O(\lambda^2) , \quad (\text{B.10a})$$

$$u_2(\alpha) = \frac{w}{1 + 2w\alpha} + \frac{\lambda\alpha}{(1 + 2w\alpha)^3} + O(\lambda^2) , \quad (\text{B.10b})$$

$$u_4(\alpha) = \frac{\lambda}{(1 + 2w\alpha)^4} + O(\lambda^2) . \quad (\text{B.10c})$$

According to (B.10b) $u_2(\alpha)$ tends to vanish as α increases. This could mean trouble since the integrand becomes flatter and flatter (this is apparent in eq.(B.6) also). The reliability of the steepest descent approximation would become increasingly doubtful. Fortunately we are saved by (B.10c) which shows that the “interaction” u_4 vanishes much faster and the integrand approaches a Gaussian, a rather flat one but still a Gaussian. The perturbative correction u_4/u_2^2 in eq.(B.9) asymptotically vanishes.

Given the vertices (B.10) correct to $O(\lambda)$ the best approximation is obtained letting $\alpha \rightarrow \infty$, which gives

$$z \approx \left(\frac{2\pi}{w}\right)^{1/2} \exp -\frac{\lambda}{8w^2}, \quad (\text{B.11})$$

which has the typical exponential of RG improved calculations.

It is quite remarkable that one can exhibit the powerful RG techniques in such a simple example. it is perhaps even more remarkable that in this simple study even their limitations become apparent. For the toy model (B.7) the exact result is known and for strong coupling (large λ) the correct dependence on λ is the power law

$$z \approx \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \left(\frac{6}{\lambda}\right)^{1/4},$$

and not the exponential dependence of (B.11). This illustrates the known fact that while RG perturbation expansions are an improvement over plain perturbation expansions, they remain nevertheless restricted to the small coupling regime. Needless to say, this is not a limitation of the RG itself (eqs.(B.3-4) are exact) but of the perturbative solution (B.10) to the RG equation (B.4).

Appendix C. The ε Expansion

The RG system of equations (19) is greatly simplified if one realizes that the solutions of interest are not the most general solutions of the first order differential equations in which the momenta q_j are independent variables, but rather those special solutions with interesting scaling properties when all q_j 's are scaled together. The unwanted solutions can be discarded by evaluating (19) at momenta λq_j with $\lambda = e^\tau$ instead of q_j . This eliminates the partial derivatives and (19) becomes

$$\begin{aligned} \left\{ \lambda \frac{d}{d\lambda} + nd_\phi - d \right\} u_n(\lambda q_1 \dots \lambda q_n, \lambda) &= \int d\tilde{k} \dot{\alpha}(k) u_{n+2}(\lambda q_1 \dots \lambda q_n, k, -k, \lambda) + \\ &\sum_{m \geq 2}^n \binom{n}{m-1} \frac{1}{n!} \sum_{\{q_j\}} \dot{\alpha}(\lambda k_m) u_m(\lambda q_1 \dots \lambda q_{m-1}, \lambda k_m, \lambda) \\ &\times u_{n-m+2}(\lambda q_m \dots \lambda q_n, -\lambda k_m, \lambda) \end{aligned} \quad (\text{C.1})$$

Substituting eq.(20) and (21) into (C.1) leads to a set of first order ordinary differential equations for the V 's,

$$\left(\lambda \frac{d}{d\lambda} - 2\right) V_{20} = -2\dot{\alpha} V_{20}^2, \quad (C.2)$$

$$\left(\lambda \frac{d}{d\lambda} - 2\right) V_{21} + 2\gamma_1 V_{20} = \int d\tilde{k} \dot{\alpha} V_{41} - 4\dot{\alpha} V_{20} V_{21}, \quad (C.3)$$

$$\begin{aligned} \left(\lambda \frac{d}{d\lambda} - 2\right) V_{22} + 2\gamma_1 V_{21} + 2\gamma_2 V_{20} &= \int d\tilde{k} \dot{\alpha} V_{42} + \frac{d}{d\varepsilon} \int d\tilde{k} \dot{\alpha} V_{41} \Big|_{\varepsilon=0} \\ &\quad - 2\dot{\alpha} V_{21}^2 - 4\dot{\alpha} V_{20} V_{22} \end{aligned} \quad (C.4)$$

$$\lambda \frac{d}{d\lambda} V_{41} = -2\left(\sum_j^4 \dot{\alpha} V_{20}\right) V_{41}, \quad (C.5)$$

$$\lambda \frac{d}{d\lambda} V_{42} + (4\gamma_1 - 1) V_{41} = \int d\tilde{k} \dot{\alpha} V_{62} - 2\left(\sum_j^4 \dot{\alpha} V_{20}\right) V_{42} - 2\left(\sum_j^4 \dot{\alpha} V_{21}\right) V_{41}, \quad (C.6)$$

$$\left(\lambda \frac{d}{d\lambda} - 2\right) V_{62} = -2\left(\sum_j^6 \dot{\alpha} V_{20}\right) V_{62} - 2(\dot{\alpha} V_{41} V_{41} + \text{perm.}). \quad (C.7)$$

The arguments of the V 's can be easily obtained by referring back to eq.(C.1). In eq.(C.7) the ten permutations refer to the inequivalent ways of grouping six momenta into two sets of three.

Equation (C.2) is of the Bernoulli type. The solution behaving as $V_{20} \approx q^2$ for small q is

$$V_{20}(q) = q^2 f(q),$$

where

$$f(q) = \frac{1}{1 + q^2 \int_0^1 d\lambda^2 \dot{\alpha}(\lambda q)} = \exp -2 \int_0^1 \frac{d\lambda}{\lambda} \dot{\alpha}(\lambda q) V_{20}(\lambda q). \quad (C.8)$$

The second equality is very useful because it will allow us to construct integrating factors for all the remaining eqs.(C3-7) which are linear. The solution to (C.5) is

$$V_{41}(q_1 \dots q_4) = A \prod_{j=1}^4 f(q_j).$$

Next we solve (C.3). The fixed point ($\partial V_{21}/\partial \lambda = 0$) and the analyticity requirements force us to choose $\gamma_1 = 0$ so that

$$V_{20}(q) = (Cq^2 - \frac{1}{2}AB)f^2(q),$$

where C is a constant and

$$B = \int d\tilde{k} \dot{\alpha}(k) f^2(k). \quad (C.9)$$

This completes the solution to order ε .

The solution for (C.7) is straightforward,

$$V_{62}(q_1 \dots q_4) = -A^2 \left[\prod_{j=1}^4 f(q_j) \right] [D(q_1 + q_2 + q_3) + \text{perm.}] , \quad (\text{C.10})$$

where

$$D(k) = \frac{1}{k^2} [1 - f(k)] . \quad (26)$$

The solution for V_{42} is messier; the only important point being that in order to eliminate a divergence at $\lambda = 0$ (or $q = 0$) the constant A in V_{41} must be chosen to be $A = (4\pi)^2/3$.

Finally, we turn to V_{22} . Again its explicit form is not very illuminating but the requirement that it be analytic at $q = 0$ determines both $V_{42}(0 \dots 0)$ which is not in itself very interesting and also γ_2 ,

$$\gamma_2 = -\frac{(4\pi)^4}{18} \int d\tilde{k} d\tilde{k}' \frac{1}{k^2} f(k) f'(k') f''(Q) , \quad (\text{C.11})$$

where $\vec{Q} = \vec{k} + \vec{k}'$ and $f' = df(k)/dk^2$. The integral in eq.(C.11) can be done analytically for $\dot{\alpha} = \Lambda^{-2} \exp(q^2/\Lambda^2)$, or else numerically for $\dot{\alpha} = \Lambda^{-2}$ or for $\dot{\alpha} = q^2/\Lambda^4$. The result is $\gamma_2 = 1/108$ so that

$$\gamma_\phi = \frac{\varepsilon^2}{108}$$

as it should be [2]. This completes the solution to order ε^2 .

Appendix D. The Perturbative Solution

Substituting eq.(22) and (23) into (C.1) leads to a set of first order ordinary differential equations for the U 's,

$$\left(\lambda \frac{d}{d\lambda} - 2 \right) U_{20} = -2\dot{\alpha} U_{20}^2 , \quad (\text{D.1})$$

$$\begin{aligned} \left(\lambda \frac{d}{d\lambda} - 2 + b_1 \right) U_{21} + 2\gamma_1 U_{20} &= \Lambda^\varepsilon \int d\tilde{k} \dot{\alpha} U_{41} \\ &\quad - 4\dot{\alpha} U_{20} U_{21} , \end{aligned} \quad (\text{D.2})$$

$$\begin{aligned} \left(\lambda \frac{d}{d\lambda} - 2 + 2b_1 \right) U_{22} + (b_2 + 2\gamma_1) U_{21} + 2\gamma_2 U_{20} &= \Lambda^\varepsilon \int d\tilde{k} \dot{\alpha} U_{42} \\ &\quad - 2\dot{\alpha} U_{21}^2 - 4\dot{\alpha} U_{20} U_{22} \end{aligned} \quad (\text{D.3})$$

$$\left(\lambda \frac{d}{d\lambda} + b_1 - \varepsilon\right) U_{41} = -2\left(\sum_j^4 \dot{\alpha} U_{20}\right) U_{41} , \quad (\text{D.4})$$

$$\begin{aligned} \left(\lambda \frac{d}{d\lambda} + 2b_1 - \varepsilon\right) U_{42} + (b_2 + 4\gamma_1) U_{41} &= \Lambda^\varepsilon \int d\tilde{k} \dot{\alpha} U_{62} - 2\left(\sum_j^4 \dot{\alpha} U_{20}\right) U_{42} \\ &\quad - 2\left(\sum_j^4 \dot{\alpha} U_{21}\right) U_{41} , \end{aligned} \quad (\text{D.5})$$

$$\left(\lambda \frac{d}{d\lambda} + 2b_1 + 2 - 2\varepsilon\right) U_{62} = -2\left(\sum_j^6 \dot{\alpha} U_{20}\right) U_{62} - 2(\dot{\alpha} U_{41} U_{41} + \text{perm.}) . \quad (\text{D.6})$$

These equations are naturally very similar to those of appendix C and the solution proceeds exactly as before except that now, as discussed in section 5, no requirement of analyticity is made.

The solution of (D.1) behaving as $U_{20} \approx q^2$ for small q is

$$U_{20}(q) = q^2 f(q)$$

with $f(q)$ given by (C.8) as before. The integration of (D.4) is straightforward. We want U_{41} independent of λ so that all the evolution of gU_{41} is attributed to the evolution of g . This forces us to choose $b_1 = \varepsilon$. Further we normalize $U_{41}(0 \dots 0) = 1$ which is the conventional normalization for g . Then

$$U_{41}(q_1 \dots q_4) = \prod_{j=1}^4 f(q_j) .$$

Next consider (D.2). The requirement that U_{21} is independent of λ implies $\gamma_1 = 0$ and one obtains the generally non-analytic expression

$$U_{21}(q) = \Lambda^\varepsilon \left(C q^{2-\varepsilon} - \frac{B}{2-\varepsilon} \right) f^2(q) ,$$

where B is given by (C.9). The solution of U_{62} is uneventful; one obtains the expression (C.10) with $A = 1$. Finally, the solution of (D.5) for U_{42} does not depend on λ provided one chooses $b_2 = -3/(4\pi)^2 + O(\varepsilon)$, that is

$$-\frac{dg}{d\tau} = \beta(g) = -\varepsilon g + \left(\frac{3}{(4\pi)^2} + O(\varepsilon) \right) g^2 + \dots$$

as expected. We stop here since the solution for U_{22} and γ_2 proceeds in the same way.

One final comment concerning the choice of “cutoff” function $\dot{\alpha}$: if one chooses a constant $\dot{\alpha} = \Lambda^{-2}$ the vertex functions develop divergences as $d \rightarrow 4$. This is evident when one computes the constant B given by (C.9). As discussed further in section 5 this is not really a problem, particularly since physically significant quantities such as $\gamma_\phi(g)$ and $\beta(g)$ are perfectly finite and independent of $\dot{\alpha}$.

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